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VANISHING LOCAL SCALAR INVARIANTS ON GENERALIZED PLANE WAVE MANIFOLDS

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ABSTRACT

We study the structure group of a canonical algebraic curvature tensor built from a symmetric bilinear form, and show that in most cases it coincides with the isometry group of the symmetric form from which it is built. Our main result is that the structure group of the direct sum of such canonical algebraic curvature tensors on a decomposable model space must permute the subspaces Vi on which they are defined. For such an algebraic curvature tensor, we show that if the vector space V is a direct sum of subspaces V1 and V2, the corresponding structure group decomposes as well if V1 and V2 are invariant of the action of the structure group on V.

1.0 INTRODUCTION

For example, an understanding of the Osserman conjecture in the higher signature setting is concerned with an algebraic understanding of the Jordan normal form of the Jacobi operator, Stanilov-Tsankov theory is concerned with the commutativity of certain other natural operators associated to the Riemann curvature tensor, and other authors have studied certain algebraic questions concerning these algebraic curvature tensors simply because these questions are of interest in their own right. Several examples of this include work on the algebraic properties of the Jacobi operator on complex model spaces, the study of the linear independence of certain sets of algebraic curvature tensors, and results aimed at improving the efficiency with which one may express a given algebraic curvature tensors.

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Let α1, . . . , αn be a collection of contra variant tensors on V . We call the tuple M := (V, α1, . . . , αn) a model space. For example, if ϕ is a symmetric bilinear form, and R is an algebraic curvature tensor, (V, ϕ, R) is a model space. There are some places in the literature where it has been convenient to distinguish certain types of model spaces from others. For example, in the pair (V, R) is referred to as a weak model space, although in the current work it is not necessary to make this distinction.

$$G_{\mathfrak{M}} = \{A \in Gl(V) | A^* \alpha_i = \alpha_i \text{ for } i = 1, \ldots, n\}$$

In the event that n = 1 so that the model space M = (V, α), then we sometimes write GM = Gα for simplicity when there is no confusion as to what is meant. In addition, we may also refer to Gα as the structure group of α for simplicity, rather than as the structure group of the model space (V, α). Structure groups arise under different names in situations that are familiar to mathematicians.

Let V be a real vector space of finite dimension N, let V ∗ := Hom(V, R) be its dual. An object R ∈ ⊗4V ∗ is called an algebraic curvature tensor if it satisfies the following three properties, the last of which is known as the Bianchi identity:

$$\begin{aligned} R(x,y,z,w) &= -R(y,x,z,w), \\ R(x,y,z,w) &= R(z,w,x,y), \text{ and} \\ 0 &= R(x,y,z,w) + R(x,z,w,y) \\ &\quad + R(x,w,y,z)\,. \end{aligned}$$

If (M, g) is a pseudo-Riemannian manifold, then one may use the Levi-Civita connection ∇ to compute the Riemann curvature tensor R∇ ∈ ⊗4T ∗M, and the evaluation of this tensor at a point P ∈ M produces the algebraic curvature tensor R∇ P ∈ ⊗4T ∗ PM, where T ∗M is the cotangent bundle of M, and T ∗ PM is the cotangent space of M at P. It is a classical differential geometric fact that every algebraic curvature tensor R can be realized as the curvature tensor of a pseudo Riemannian manifold at a point. Thus it can be said that these algebraic curvature tensors are an algebraic portrait of the curvature of a manifold at a point, and an understanding of these algebraic objects often translates into a subsequent understanding of the geometrical object they represent.

For example, if ϕ is a positive-definite inner product, then Gϕ = O(N), the familiar orthogonal group. If one notes that Gl(V) is the structure group of the trivial model space consisting solely of V , then many important quantities are dependent upon the observation that the quantity they compute be "independent of the particular basis chosen," the determinant and trace of a linear operator, for example

2.0 LITERATURE REVIEW

E. Abbena, S. Garbiero, L. Vanhecke (1992) A direct, bundle-theoretic method for defining and extending local isometries out of curvature data is developed. As a by-product, conceptual direct proofs of a classical result of Singer and a recent result of the authors are derived. A classical result of I. M. Singer states that a Riemannian manifold is locally homogeneous if and only if its Riemannian curvature tensor together with its covariant derivatives up to some index k+1 are independent of the point (the integer k is called the Singer invariant). More precisely Theorem 1. Let M be a Riemannian manifold. Then M is locally homogeneous if and only if for any p, q ∈ M there is a linear isometry F: TpM → TqM such that F ∗ ∇ s Rq = ∇ s Rp, for any s ≤ k + 1. An alternate proof with a more direct approach was given.

D. V. Alekseevsky, A. S. Galaev (2011) Out of the curvature tensor and its covariant derivatives one can construct scalar invariants, like for instance the scalar curvature. In general, any polynomial function in the components of the curvature tensor and its covariant derivatives which does not depend on the choice of the orthonormal basis at the tangent space of each point is a a scalar Weyl invariant or a scalar curvature invariant. By Weyl theory of invariants, a scalar Weyl invariant is a linear combination of complete traces of tensors 〈 ∇ m1 R, 〉 ... 〈 ∇ mℓ R, 〉 , (m1,...ms ≥ 0, ∇ 0 R = R). Prüfer, Tricerri and Vanhecke studied the interplay among local homogeneity and these curvature invariants. Using Singer's Theorem, they got the following Theorem 2 (Prüfer, Tricerri and Vanhecke [6]). Let M be an n-dimensional Riemannian manifold. Then M is locally homogeneous if and only if all scalar Weyl invariants of order s with s ≤ n(n−1) 2 are constant. More in general for a non-homogeneous Riemannian manifold one can look at the regular level sets of scalar Weyl invariants.

O. F. Blanco, M. S´anchez, J. M. Senovilla (2010) We construct a new family of curvature homogeneous pseudo-Riemannian manifolds modeled on $\mathbb{R}^{3k+2}$ for integers $k \geq 1$. In contrast to previously known examples, the signature may be chosen to be $(k+1+a, k+1+b)$ where $a,b \in \mathbb{N} \bigcup \{0\}$ and $a+b = k$. The structure group of the 0-model of this family is studied, and is shown to be indecomposable. Several invariants that are not of Weyl type are found which will show that, in general, the members of this family are not locally homogeneous.

M. Blau, M. O'Loughlin (2003) We show that generalized plane wave manifolds are complete, strongly geodesically convex, Osserman, Szab´o, and Ivanov–Petrova. We show their holonomy groups are nilpotent and that all the local Weyl scalar invariants of these manifolds vanish. We construct isometry invariants on certain families of these manifolds which are not of Weyl type. Given k, we exhibit manifolds of this type which are k-curvature homogeneous but not locally homogeneous. We also construct a manifold which is weakly 1-curvature homogeneous but not 1-curvature homogeneous.

3.0 METHODOLOGY

The first step for study the geometry of (pseudo-)Riemannian manifolds is to determine the Lievi-Civita connection. By using the Koszul identity,

$$\begin{aligned}2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),\end{aligned}$$

and applying the metric (1.2), one can determine the components of the LeviCivita connection. We use ∂i = ∂ ∂xi as a local basis for the tangent space and have:

Theorem 1

Let (M, g) be an arbitrary two-symmetric Lorentzian fourmanifold, where the metric g is described in local coordinates (x 1 , x2 , x3 , x4) by the Equation (1.2). The non-zero components of the Levi-Civita connection are:

$$\begin{aligned}&\nabla_{\partial_2}\partial_4 = (ax^2x^4 + px^2 + qx^3)\partial_1,\\ &\nabla_{\partial_3}\partial_4 = (bx^3x^4 + sx^3 + qx^2)\partial_1,\\ &\nabla_{\partial_4}\partial_4 = \tfrac{a(x^2)^2 + b(x^3)^2}{2}\partial_1\\ &-(ax^2x^4 + px^2 + qx^3)\partial_2 - (bx^3x^4 + qx^2 + sx^3)\partial_3.\end{aligned}$$

Applying the relation R(X, Y) = [∇X, ∇Y] − ∇[X,Y] we immediately determine the curvature tensor. If we set R(∂k, ∂l)∂j = Ri jkl∂i , then by contraction on the first and third indices of the curvature tensor, the Ricci tensor ϱ will be deduced. The scalar curvature tensor τ is also obtained by full contraction of coefficients of the curvature tensor

Theorem 2

A four-dimensional two-symmetric Lorentzian space admits zero scalar curvature. Also, the non-zero components of curvature tensor and Ricci tensor are:

$$\begin{aligned}&R(\partial_2,\partial_4) = (ax^4+p)\partial_1 dx^2 + q\partial_1 dx^3 - (ax^4+p)\partial_2 dx^4 - q\partial_3 dx^4,\\ &R(\partial_3,\partial_4) = q\partial_1 dx^2 + (bx^4+s)\partial_1 dx^3 - q\partial_2 dx^4 - (bx^4+s)\partial_3 dx^4,\\ &\varrho(\partial_4,\partial_4) = -(a+b)x^4 - (s+p).\end{aligned}$$

A (pseudo-)Riemannian manifold (M, g) is called Einstein if ϱ = cg, for a real constant c. Being Ricci flat means that the Ricci tensor vanishes identically. Also, conformal flatness translates into the following system of algebraic equations:

$$W_{ijkh} = R_{ijkh} - \frac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{ih}\varrho_{jk} - g_{jk}\varrho_{ih})$$
$$+\frac{\tau}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}) = 0 \quad \text{for all indices } i, j, k, h = 1, \ldots, 4,$$

where W denotes the Weyl tensor and τ is the scalar curvature. Although two-symmetric spaces clearly aren't flat, we can check Ricci flatness.

4.0 RESULTS

Let Mf and M be defined as above, and let p ≥ 3. Assume that the Hessian H of f has rank p and has constant signature. Let ϕ ∈ S 2 (V) have the same (constant) signature as H.

(1) The curvature tensor R and its covariant derivative ∇R of Mf satisfies (a) R = RH. (b) ∇R(Z1, Z2, Z3, Z4;Z5) = Z5(R(Z1, Z2, Z3, Z4)). (c) kerR = ker ∇R = span{∂y1 , . . . , ∂yp }.

(2) Mf is curvature homogeneous with model M.

(3) Mf is a generalized plane wave manifold. Thus, all Weyl scalar invariants vanish, and Mf is complete.

(4) Suppose H and ϕ have signature (r, s). If f = 1 2 (−x 2 1 − · · · − x 2 r + x 2 r+1 + · · ·+x 2 r+s), then Mf satisfies ∇R = 0; that is, Mf is symmetric and hence locally homogeneous. Thus, M is the model of a symmetric space.

(5) For generic choices of f, Mf is not locally homogeneous.

The following is a construction of an isometry invariant αf of Mf that proves Assertion (5) of Theorem 5.6; this invariant was originally constructed. We use the language and results of this paper to rephrase this construction in an effort to keep the paper self-contained. Define the model space M¯ P = (V , ¯ R, ¯ A¯), where the elements in this model space are as follows:

$$\begin{aligned} \bar{V} &= T_P M/(\ker R|_P), \text{ with } \pi : T_P M \to \bar{V} \text{ the natural projection.} \\ R|_P = (R_H)|_P &= \pi^* \bar{R}. \\ \nabla R|_P &= \pi^* \bar{A}. \end{aligned}$$

One notices that R¯ = Rϕ¯, where H = π ∗ϕ¯, and according to Theorem, the structure group of the model space (V , ¯ R¯) is the group of linear transformations A that have A∗ϕ¯ = ±ϕ¯ or A∗ϕ¯ = ¯ϕ, depending on the signature of ¯ϕ. One then defines αf as the absolute value of the square length of A¯ with respect to ¯ϕ. Specifically, if {X1, . . . , Xp} is a basis for V¯ that is orthonormal with respect to ¯ϕ, then,

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$$\alpha_f := \left| \sum_{ijk\ell n} \bar{\varphi}(X_i, X_i)\bar{\varphi}(X_j, X_j)\bar{\varphi}(X_k, X_k)\bar{\varphi}(X_\ell, X_\ell)\bar{\varphi}(X_n, X_n)\bar{A}(X_i, X_j, X_k, X_\ell; X_n)^2 \right|.$$

Since any change of orthonormal basis with respect to ¯φ preserves αf , it is invariant under the action of the structure group of (V , ¯ R¯) (this is the purpose of the absolute value above, in the event that A∗φ¯ = −φ¯), and hence an invariant of M¯ P . Since αf is built from quantities that are preserved by isometry, αf is an isometry invariant. One may show that for generic f, the invariant αf is not constant, and hence in this case Mf is not locally homogeneous. In particular, αf is an invariant of Mf which is not of Weyl type, otherwise it would vanish according to Assertion (3) of Theorem.

See for more on this invariant. A similar invariant is constructed in the signature (2, 2) case, see or [3], or [8] for somewhat different approaches to constructing this invariant–both rely on an understanding of the structure group of (V, Rφ) for their construction. We conclude this section and construct an example that realizes the model space in Corollary 1.8, and illustrates the use of Theorem 1.11 in the geometric setting.

To each element of the structure group GR for R = ⨁k i=1Rφi , there is a permutation σ with A : Vi → Vσ(i) . We apply these results to various situations of broad interest in Section 4 that help us to classify, up to group isomorphism, the structure group GR when R ∈ A(V) is the direct sum of canonical algebraic curvature tensors. We show that if there are ever two subspaces of V which are invariant by the action of the structure group, then the structure group itself decomposes as an internal direct product. In this case, the decomposition of the model space gives rise to a decomposition of the structure group. Such a situation arises if, for example if V = ⨁k i=1Vi , the subspaces Vi have different dimensions. In the event the subspaces Vi have the same dimension and the forms φi all have the same signature (or reversed signature), then the structure group can be recovered entirely from this data as the wreath product of GRφ (which has been computed), and the full symmetric group Sk.

We also show that in the event the subspaces Vi and Vj share the same dimension but φi and φj have incompatible signatures, then any element of the structure group must not permute Vi to Vj . We close our study by noting that combinations of these results are also possible, and these combinations allow one to determine the (group) isomorphism class of GR. We finish our study by describing how one actually would do this in practice. Let (V, R) = ⨁k i=1(Vi , Rφi). Since Rφi = R−φi , exchange if necessary φi with −φi to force the signature (pi , qi) to satisfy pi ≤ qi .

Then partition the Vi according to the signatures of these ϕi defined on each Vi . If one direct sums each (Vi , Rϕi) making up any partition, then the structure group of the resulting model space will be a wreath product of Gϕ by some (full) symmetric group, where ϕ is any one of the forms put in this partition. According to Corollary 1.8, each of the new model spaces on each partition are modeled on vector subspaces which are GRϕ −invariant. According to Corollary 1.7, the resulting structure group GR will be isomorphic to a direct product of wreath products.

5.0 CONCLUSION

We apply these results to various situations of broad interest in Section 4 that help us to classify, up to group isomorphism, the structure group GR when R ∈ A(V) is the direct sum of canonical algebraic curvature tensors. We show that if there are ever two subspaces of V which are invariant by the action of the structure group, then the structure group itself decomposes as an internal direct product. In this case, the decomposition of the model space gives rise to a decomposition of the structure group. Such a situation arises if, for example if V = ⨁k i=1Vi , the subspaces Vi have different dimensions. In the event the subspaces Vi have the same dimension and the forms ϕi all have the same signature (or reversed signature), then the structure group can be recovered entirely from this data as the wreath product of GRϕ (which has been computed already), and the full symmetric group Sk. We also show that in the event the subspaces Vi and Vj share the same dimension but ϕi and ϕj have incompatible signatures, then any element of the structure group must not permute Vi to Vj . We close our study by noting that combinations of these results are also possible, and these combinations allow one to determine the (group) isomorphism class of GR. We finish our study by describing how one actually would do this in practice.

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