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THREE DIMENSIONAL WAVES AT THE INTERFACE OF TWO LIQUIDS- EACH OF FINITE DEPTH

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***Abstract*— A method essentially based on a simple reduction procedure is employed here to find the linear solution for the obliquely incident incoming waves at the interface of two superposed liquid against a vertical cliff. Solution of the problem is obtained here assuming no reflection of waves by the cliff and both the liquids to be of finite constant depth. In the absence of the upper liquid corresponding result for a single liquid is recovered and identified with the known result.**

***Keywords*— incoming wave, inviscid liquid, irrotational flow, velocity potential, linear theory, vertical cliff**

1. INTRODUCTION

The intention of the present study is to find the linear solution for the three-dimensional problem of incoming waves, at the interface of two superposed liquid, assuming that the liquids are bounded on the left by a rigid vertical cliff. If the reflection of waves by the cliff is neglected, then it follows that the cliff bound wave carries certain energy with it and is totally reflected back, as there is no mechanism to absorb the incoming energy in an inviscid fluid system ([1]).

The problem containing a vertical cliff in a single fluid medium is a subject of considerable research interest among scientists and engineers, as the problem is a special case of the well known sloping beach problem ([2]-[9]).

But the Problems involving two superposed liquid separated by a common interface have not received much care as the problems are, in general, difficult because of the coupled boundary conditions at the interface of the two liquids ([10]-[12]).

The present study is related with the three-dimensional problem of incoming wave progressing towards a rigid vertical cliff, in two immiscible liquid, where the lower liquid is of finite constant depth h and the upper liquid is of finite constant height H. Allowing no reflection of waves by the cliff, the problem under consideration is attacked for solution, assuming linear theory, by a simple reduction procedure and analytical expressions for the velocity potentials, in each of the two liquid, are obtained. In the absence of the upper liquid, corresponding result for a single liquid is recovered and identified with the known result ([9]).

2. STATEMENT OF THE PROBLEM

Let us consider the three-dimensional irrotational motion of two inviscid, homogeneous liquid with densities ρ_1 and ρ_2 ($< \rho_1$), of the lower and upper liquid respectively, under the action of gravity. A rectangular cartesian co-ordinate system is chosen in which the y-axis is taken to be vertically downwards into the lower liquid, the plane y=0, x > 0 is the mean position of the interface, x = 0 is the rigid cliff, and the two liquid of densities ρ_1 and ρ_2 occupy the regions $x > 0$, $0 < y \leq h$, $-\infty < z < \infty$ and $x > 0$, $-H \leq y < 0$, $-\infty < z < \infty$ respectively. The origin is taken at a point on the line of intersection where the mean interface and the cliff meet.

3. FORMULATION OF THE PROBLEM

The problem is to find the velocity potentials $\Phi_j(x, y, z, t)$, with $j = 1,2$ ($j = 1$ for the lower liquid and $j = 2$ for the upper liquid) which represent progressive waves moving towards the shore line (i.e. the z-axis) such that the wave crests at large distance from the shore tend to straight lines which make an arbitrary angle θ with the shore line. For periodic motion, we can assume

$$\Phi_j(x, y, z, t) = Re\left[\, \phi_j(x, y) exp\{-i(\sigma t + \nu z)\}\right]$$

(1)

where $\nu = \alpha_0 \sin\theta$ and α_0 is the unique positive real root of the transcendental equation

$$K(\coth kh + s \coth kH = k(1 - s)$$

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and σ is the angular frequency. The problem is to find the solution for Φ_1, Φ_2 which behave, as $x \to \infty$ like progressive wave moving towards the cliff.

Assuming linear theory, the potential functions $\phi_1(x,y)$ and $\phi_2(x,y)$ satisfy the following boundary value problems:

(i) Two-dimensional modified Helmholtz's equations :

$$(\nabla^2 - \nu^2)\phi_j = 0 \qquad (2)$$

in the respective flow domain, where ∇^2 is the two dimensional Laplacian.

(ii) Linearized form of the interface conditions on $y = 0, x > 0$

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial y} &= \frac{\partial\phi_2}{\partial y} \\ K\phi_1 + \frac{\partial\phi_1}{\partial y} &= s\left(K\phi_2 + \frac{\partial\phi_2}{\partial y}\right) \end{aligned}\right\} \qquad (3)$$

where $K = \sigma^2/g$ is the wave number, g is the acceleration due to gravity and $s = \rho_2/\rho_1$.

(iii) Conditions at the rigid bottom and top:

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial y} &= 0, \quad on\ \ y = h \\ \frac{\partial\phi_2}{\partial y} &= 0, \quad on\ \ y = -H \end{aligned}\right\} \quad x > 0 \qquad (4)$$

(iv) Conditions on the rigid cliff $x = 0$:

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial x} &= 0, \quad 0 < y \le h \\ \frac{\partial\phi_2}{\partial x} &= 0, \quad -H \le y < 0 \end{aligned}\right\} \qquad (5)$$

Our purpose is to obtain ϕ_1, ϕ_2 satisfying (2) to (5) and the condition that they behave at infinity as progressive waves moving towards the cliff.

Further, as no reflection of waves, by the cliff, is allowed which can be justified by assuming a source/sink type behavior in the potential functions at the origin, in the absence of surface tension ([4]), which lead to the conditions

$$\phi_1, \phi_2 \sim \ln r \quad as\ \ r = (x^2+y^2)^{1/2} \to 0. \qquad (6)$$

Noting the conditions (3) and (4), and following Gorgui and Kassem [13], we can assume that as $x \to \infty$:

$$\left.\begin{aligned} \phi_1 &\sim \frac{\cosh \alpha_0(h-y)}{\sinh \alpha_0 h}\exp(-i\mu x) \\ \phi_2 &\sim -\frac{\cosh \alpha_0(H+y)}{\sinh \alpha_0 H}\exp(-i\mu x) \end{aligned}\right\} \qquad (7)$$

where $\mu = \alpha_0 \cos\theta$.

4. SOLUTION OF THE PROBLEM

To solve the problem mathematically we reduce the boundary value problem, described by (2) to (6) and the infinity requirement given by (7), to another boundary value problem. To do this, let us introduce two new functions ψ_1 and ψ_2 of (x,y) by the following relations:

$$\left.\begin{aligned} \phi_1 &= \frac{2\cosh \alpha_0(h-y)}{\sinh \alpha_0 h}\cos\mu x + \psi_1 \\ \phi_2 &= -\frac{2\cosh \alpha_0(H+y)}{\sinh \alpha_0 H}\cos\mu x + \psi_2 \end{aligned}\right\} \qquad (8)$$

where ψ_1, ψ_2 satisfy the boundary value problems described by (2) to (6) together with the infinity requirements (as $x \to \infty$):

$$\left.\begin{aligned} \psi_1 &\to -\frac{\cosh \alpha_0(h-y)}{\sinh \alpha_0 h}\exp(i\mu x) \\ \psi_2 &\to \frac{\cosh \alpha_0(H+y)}{\sinh \alpha_0 H}\exp(i\mu x) \end{aligned}\right\} \qquad (9)$$

It should be mentioned here that, as $x \to \infty$, ψ_1, ψ_2 defined by (8) represent outgoing wave, however, ϕ_1, ϕ_2 represent incoming wave, though ψ_1, ψ_2 and ϕ_1, ϕ_2 satisfy the same boundary value problems described by (2) to (6). Thus if ψ_1, ψ_2 are known, the
time independent potential functions ϕ_1, ϕ_2 can be derived by using (8).

Alternative representation for ψ_1, ψ_2 satisfying (2) to (6), are given by

$$\left.\begin{aligned} &\psi_1(x,y) = \\ &c\oint_{\alpha_0}^{\infty} \frac{k\cosh k(h-y)\sinh kH}{\delta(k)}\frac{\cos\eta x}{\eta}dk \\ &\psi_2(x,y) = \\ &-c\oint_{\alpha_0}^{\infty} \frac{k\cosh k(H+y)\sinh kh}{\delta(k)}\frac{\cos\eta x}{\eta}dk \end{aligned}\right\} \qquad (10)$$

where $\eta = (k^2 - \upsilon^2)^{1/2}$ and
$\delta(k) =$

$k(1-s)\sinh kh \sinh kH - K \times$
$(\cosh kh \sinh kH + s \cosh kH \sinh kh)$.

Here $\delta(k)$ has a simple pole at $k=\alpha_0>0$ (say), a simple pole at $k=\alpha' < 0$ (say), and an infinite number of complex poles with positive real part, of the form $s\xi_n \pm i\alpha_n$ ([10]). Here the contour is indented below the pole at $k=\alpha_0$ to account for the outgoing nature of ψ_1, ψ_2 as $x\to\infty$, and c is a constant to be determined such that the conditions at infinity given by (9) are satisfied. It can be easily shown that ψ_1, ψ_2 given by (10) satisfy the interface conditions (3) and for small r they behave like $\ln r$ ([4]). It is to be noted here that in the absence of the upper liquid (i.e when $s=0$), $\delta(k)=0$ has a simple real root $\alpha_0>0$, and an infinite number of purely imaginary roots of the form $\pm i\alpha_n$(cf. [14]).
ψ_1, ψ_2 given by (10), may also be represented as follows:

$$\left.\begin{aligned}
&\psi_1(x,y) = \\
&\tfrac{2\pi i c}{1-s}\Big[\sec\theta \tfrac{g(\alpha_0)}{f(\alpha_0)}\cosh\alpha_0(h-y) \\
&\times \sinh\alpha_0 H \exp(i\mu x) + \textstyle\sum \gamma \tfrac{g(\gamma)}{f(\gamma)} \\
&\times \cosh\gamma(h-y)\sinh\gamma H \tfrac{\exp(i\mu_1 x)}{\mu_1} \\
&-\textstyle\sum\bar\gamma \tfrac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma(h-y)\sinh\bar\gamma H \\
&\times\tfrac{\exp(-i\mu_2 x)}{\mu_2}\Big] \\
&\psi_2(x,y) = \\
&-\tfrac{2\pi i c}{1-s}\Big[\sec\theta \tfrac{g(\alpha_0)}{f(\alpha_0)}\cosh\alpha_0(H+y) \\
&\times \sinh\alpha_0 h \exp(i\mu x) + \textstyle\sum \gamma \tfrac{g(\gamma)}{f(\gamma)} \\
&\times \cosh\gamma(H+y)\sinh\gamma h \tfrac{\exp(i\mu_1 x)}{\mu_1} \\
&-\textstyle\sum\bar\gamma \tfrac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma(H+y)\sinh\bar\gamma h \\
&\times\tfrac{\exp(-i\mu_2 x)}{\mu_2}\Big]
\end{aligned}\right\} \tag{11}$$

where
$f(x) =$
$(2xh+\sinh 2xh)\sinh^2 xH + s\sinh^2 xh\,(2xH + \sinh 2xH)$,

$g(x) =$
$\cosh xh \sinh xH + s\cosh xH \sinh xh$,
$\gamma = s\xi_n + i\alpha_n,\ \bar\gamma = s\xi_n - i\alpha_n,\ n=1,2,3,\ldots.$
$\mu_1 = (\gamma^2-\upsilon^2)^{1/2},\ \mu_2 = (\bar\gamma^2-\upsilon^2)^{1/2}$.

Infinity requirements, given by (9), are satisfied by choosing

$$c = i\frac{(1-s)D}{2\pi} \tag{12}$$

where
$D = \cos\theta\left[\frac{g(\alpha_0)}{f(\alpha_0)}\sinh\alpha_0 h \sinh\alpha_0 H\right]^{-1}$.

Using (12) into (11), the functions ψ_1, ψ_2 can be found, which are given by

$$\left.\begin{aligned}
&\psi_1(x,y) = \\
&-\tfrac{\cosh\alpha_0(h-y)}{\sinh\alpha_0 h}\exp(i\mu x) - D\textstyle\sum\gamma\tfrac{g(\gamma)}{f(\gamma)} \\
&\times\cosh\gamma(h-y)\sinh\gamma H\tfrac{\exp(i\mu_1 x)}{\mu_1} \\
&+D\textstyle\sum\bar\gamma\tfrac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma(h-y)\sinh\bar\gamma H \\
&\times\tfrac{\exp(-i\mu_2 x)}{\mu_2}\Big] \\
&\psi_2(x,y) = \\
&\tfrac{\cosh\alpha_0(H+y)}{\sinh\alpha_0 H}\exp(i\mu x) + D\textstyle\sum\gamma\tfrac{g(\gamma)}{f(\gamma)} \\
&\times\cosh\gamma(H+y)\sinh\gamma h\tfrac{\exp(i\mu_1 x)}{\mu_1} \\
&-D\textstyle\sum\bar\gamma\tfrac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma(H+y)\sinh\bar\gamma h \\
&\times\tfrac{\exp(-i\mu_2 x)}{\mu_2}\Big].
\end{aligned}\right\} \tag{13}$$

Exploiting (13) into (8), the solutions ϕ_1, ϕ_2 for the original boundary value problems described by (2)-(6) together with conditions at infinity given by (7), are obtained, which are given by

$$\left.\begin{aligned}
&\phi_1(x,y) = \\
&\tfrac{\cosh\alpha_0(h-y)}{\sinh\alpha_0 h}\exp(-i\mu x) - D\textstyle\sum\gamma\tfrac{g(\gamma)}{f(\gamma)} \\
&\times\cosh\gamma(h-y)\sinh\gamma H\tfrac{\exp(i\mu_1 x)}{\mu_1} \\
&+D\textstyle\sum\bar\gamma\tfrac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma(h-y)\sinh\bar\gamma H \\
&\times\tfrac{\exp(-i\mu_2 x)}{\mu_2}\Big] \\
&\phi_2(x,y) = \\
&-\tfrac{\cosh\alpha_0(H+y)}{\sinh\alpha_0 H}\exp(-i\mu x) + D\textstyle\sum\gamma\tfrac{g(\gamma)}{f(\gamma)} \\
&\times\cosh\gamma(H+y)\sinh\gamma h\tfrac{\exp(i\mu_1 x)}{\mu_1} \\
&-D\textstyle\sum\bar\gamma\tfrac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma(H+y)\sinh\bar\gamma h \\
&\times\tfrac{\exp(-i\mu_2 x)}{\mu_2}\Big].
\end{aligned}\right\} \tag{14}$$

Making use of (14) into (1), the velocity potentials $\Phi_1(x,y,z,t)$, $\Phi_2(x,y,z,t)$ have been found (see Appendix-A). The explicit expressions for Φ_1,Φ_2 are given by

$$\left.\begin{aligned}&\Phi_1(x,y,z,t)=\\&\frac{\cosh\alpha_0(h-y)}{\sinh\alpha_0 h}\cos(\mu x+\sigma t+\nu z)\\&-2D\sin(\sigma t+\nu z)\sum\omega(\alpha_n)\\&\times\exp(-q_n x)\\[1em]&\Phi_2(x,y,z,t)=\\&-\frac{\cosh\alpha_0(H+y)}{\sinh\alpha_0 H}\cos(\mu x+\sigma t+\nu z)\\&+2D\sin(\sigma t+\nu z)\sum\chi(\alpha_n)\\&\times\exp(-q_n x)\end{aligned}\right\}\quad(15)$$

Φ_1, Φ_2 represented by (15) are the velocity potentials for incoming water waves against a rigid vertical cliff in two immiscible liquid.

4. SPECIAL CASE

As a special case, if we make the assumption $s = 0$ (which leads to one-fluid medium), we find (see Appendix-B):

$$D = \frac{2(2\alpha_0 h+\sinh 2\alpha_0 h)\cos\theta}{\sinh 2\alpha_0 h},$$
$$\omega(\alpha_n) = -\frac{\alpha_n\cos\alpha_n(h-y)\cos\alpha_n h}{(2\alpha_n h+\sin 2\alpha_n h)q_n},$$

so that

$$\begin{aligned}&\Phi_1(x,y,z,t)=\\&\frac{\cosh\alpha_0(h-y)}{\sinh\alpha_0 h}\cos(\mu x+\sigma t+\nu z)\\&+\frac{4(2\alpha_0 h+\sinh 2\alpha_0 h)\cos\theta}{\sinh 2\alpha_0 h}\sin(\sigma t+\nu z)\\&\times\sum\frac{\alpha_n\cos\alpha_n(h-y)\cos\alpha_n h}{(2\alpha_n h+\sin 2\alpha_n h)}\frac{\exp(-q_n x)}{q_n}\end{aligned}\qquad(16)$$

where $q_n = (\alpha_n{}^2 + \upsilon^2)^{1/2}$.

The above expression for $\Phi_1(x,y,z,t)$ is the velocity potential for a three-dimensional progressive wave train moving towards a rigid cliff in a single liquid of uniform finite depth h. The above result is somewhat similar to that obtained by Mandal and Kundu [9].

If further we assume $s = 0$, then the expression for $\Phi_1(x,y,z,t)$ given by (16) reduces to

$$\begin{aligned}&\Phi_1(x,y,t)=\\&\frac{\cosh\alpha_0(h-y)}{\sinh\alpha_0 h}\cos(\alpha_0 x+\sigma t)\\&+\frac{4(2\alpha_0 h+\sinh 2\alpha_0 h)}{\sinh 2\alpha_0 h}\sin\sigma t\\&\times\sum\frac{\cos\alpha_n(h-y)\cos\alpha_n h}{(2\alpha_n h+\sin 2\alpha_n h)}\exp(-\alpha_n x)\end{aligned}\qquad(17)$$

which is the velocity potential for a normally incident wave train progressing towards a rigid vertical cliff in a liquid of uniform finite depth h below the free surface ([9]).

4. DISCUSSION

The three dimensional problem of incoming water waves at the interface of two superposed liquid against a vertical cliff is solved here by employing a simple reduction procedure. Explicit expression for the velocity potentials in each of the two liquid are obtained here assuming both the liquids to be of uniform finite depth. In the absence of the upper liquid various known results are recovered which are in complete agreement with those found earlier by Mandal and Kundu [9]. The major advantage of the reduction procedure described here is that the solution for the corresponding two-dimensional problem can be recovered simply by the substitution $\theta = 0$.

APPENDIX-A

$$A_1 = \frac{[P(H)R(h)-Q(H)S(h)-R(h)]}{2},$$

$$A_2 = \frac{[P(H)S(h)+Q(H)R(h)-S(h)]}{2},$$

$$B_1 = [P(H)-1]s\xi_n h - Q(H)\alpha_n h,$$
$$B_2 = [P(H)-1]\alpha_n h + Q(H)s\xi_n h,$$

$$C_1 = \frac{s[P(h)R(H)-Q(h)S(H)-R(H)]}{2},$$

$$C_2 = \frac{s[P(h)S(H)+Q(h)R(H)-S(H)]}{2},$$

$$D_1 = s[\{P(h)-1\}s\xi_n H - Q(h)\alpha_n H],$$
$$D_2 = s[\{P(h)-1\}\alpha_n H + Q(h)s\xi_n H],$$
$$E_1 = T(h)V(H) - U(h)W(H),$$
$$E_2 = T(h)W(H) + U(h)V(H),$$
$$F_1 = s[T(H)V(h) - U(H)W(h)],$$
$$F_2 = s[T(H)W(h) + U(H)V(h)],$$
$$G_1 = T(h-y)V(H) - U(h-y)W(H),$$
$$G_2 = T(h-y)W(H) + U(h-y)V(H),$$
$$p_n + iq_n = (s^2\xi_n{}^2 - \alpha_n{}^2 - \nu^2 + 2is\xi_n\alpha_n)^{1/2},$$

$$M_1 = \frac{[G_1(p_n s\xi_n + q_n\alpha_n) - G_2(p_n\alpha_n - q_n s\xi_n)}{p_n{}^2 + q_n{}^2},$$

$$M_2 = \frac{[G_1(p_n\alpha_n - q_n s\xi_n) + G_2(p_n s\xi_n + q_n\alpha_n)}{p_n{}^2 + q_n{}^2},$$

$I_1 = T(H+y)V(h) - U(H+y)W(h),$
$I_2 = T(H+y)W(h) + U(H+y)V(h),$

$N_1 = \frac{[I_1(p_n s\xi_n + q_n\alpha_n) - I_2(p_n\alpha_n - q_n s\xi_n)}{{p_n}^2 + {q_n}^2},$

$N_2 = \frac{[I_1(p_n\alpha_n - q_n s\xi_n) + I_2(p_n s\xi_n + q_n\alpha_n)}{{p_n}^2 + {q_n}^2},$

$J_1 = (N_1 \cos p_n x - N_2 \sin p_n x),$
$J_2 = (N_1 \sin p_n x + N_2 \cos p_n x),$
$P(x) = \cosh 2s\xi_n x \cos 2\alpha_n x,$
$Q(x) = \sinh 2s\xi_n x \sin 2\alpha_n x,$
$R(x) = \sinh 2s\xi_n x \cos 2\alpha_n x,$
$S(x) = \cosh 2s\xi_n x \sin 2\alpha_n x,$
$T(x) = \cosh s\xi_n x \cos \alpha_n x,$
$U(x) = \sinh s\xi_n x \sin \alpha_n x,$
$V(x) = \sinh s\xi_n x \cos \alpha_n x,$
$W(x) = \cosh s\xi_n x \sin \alpha_n x,$
$X_i = A_i + B_i + C_i + D_i, \ \ (i = 1,2)$
$Y_i = E_i + F_i, \ \ (i = 1,2)$

$\omega(\alpha_n) = \frac{[H_1X_1Y_2 + H_2X_1Y_1 - H_1X_2Y_1 + H_2X_2Y_2]}{{X_1}^2 + {X_2}^2},$

$\chi(\alpha_n) = \frac{[J_1X_1Y_2 + J_2X_1Y_1 - J_1X_2Y_1 + J_2X_2Y_2]}{{X_1}^2 + {X_2}^2}.$

APPENDIX-B

Upon substitution $s = 0$
$p_n = 0, \qquad P(x) = \cos 2\alpha_n x,$
$Q(x) = R(x) = 0, \ S(x) = \sin 2\alpha_n x,$
$A_1 = 0, \ A_2 = -\sin^2 \alpha_n H \sin 2\alpha_n h,$
$B_1 = 0, \ B_2 = -2\alpha_n h \sin^2 \alpha_n H,$
$C_1 = C_2 = D_1 = D_2 = 0, \ X_1 = 0,$
$X_2 = -(2\alpha_n h + \sin 2\alpha_n h) \sin^2 \alpha_n H,$
$T(x) = \cos \alpha_n x\,, \ U(x) = V(x) = 0,$
$W(x) = \sin \alpha_n x\,, \ E_1 = F_1 = F_2 = 0,$
$E_2 = \cos \alpha_n h \sin \alpha_n H, Y_1 = 0, Y_2 = E_2,$
$G_1 = 0, \ G_2 = \cos \alpha_n (h - y) \sin \alpha_n H,$
$M_1 = 0, \ \ M_2 = \frac{\alpha_n \cos \alpha_n (h-y) \sin \alpha_n H}{q_n},$
$H_1 = 0, \ \ H_2 = M_2,$
$\omega(\alpha_n) = -\frac{\alpha_n \cos \alpha_n (h-y) \cos \alpha_n h}{(2\alpha_n h + \sin 2\alpha_n h) q_n},$
$f(\alpha_0) = (2\alpha_0 h + \sinh 2\alpha_0 h) \sinh^2 \alpha_0 H,$
$g(\alpha_0) = \cosh \alpha_0 h \sinh \alpha_0 H,$
$D = \frac{2(2\alpha_0 h + \sinh 2\alpha_0 h) \cos \theta}{\sinh 2\alpha_0 h},$

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