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Time discretized FEM solution strategies for response evaluation of SDOF systems

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***Abstract*— In the first part of this paper, an under-damped SDOF system exhibiting free and forced vibration is analyzed by Galerkin (weighted residual) method. It is evident that the displacement response history obtained by Galerkin method is in close conformity with the analytical solution. However, the velocity and acceleration responses exhibit intermittent discontinuities. Consequently, time discretized FEM formulation is developed using temporal shape functions (C2 Continuity) ensuring the continuity of velocity and acceleration functions at temporal nodes. Results of FEM formulation with time discretization of SDOF system under free and forced vibration are presented in the second part of the paper.**

***Keywords*—Weak Formulation, Galerkin, Numerical methods, Finite element method, Global equations, Nodal solution component, weighted residual, Time discretization**

1. INTRODUCTION

Engineering analysis is carried out for structures, installations, assemblies, components and machineries before putting together their parts to find out various properties such as type of load, zone of loading, developed stresses and strains, permissible deflection, vibration properties etc. These properties can be gauged using three methods of analysis viz. a) Analytical approach, b) Experimental approach and c) Numerical methods or approximation methods. Analytical approaches are methods analyzed using formulae and mathematical expressions. They are mostly theoretical in nature. Simple geometrical objects like beams, columns, plates etc only can be analyzed by these methods. Experimental approach refers to testing of the component or its prototype by using testing equipment and assembly. Manpower and materials as well as machinery is required for Experimental method which makes it an expensive method. Numerical methods are used in items of complicated dimensions, shapes with complex material properties and boundary conditions. Numerical methods give solutions that are suitable although approximate.

The widely held problems in the field of physics, chemistry, engineering, aeronautics, mathematics and biology etc. are expressed by using differential equations. These differential equations can be ordinary or partial differential equations. These equations are then solved using various Numerical methods such as Functional Approximation Methods, Finite Element Method (FEM), Finite Volume Method, Finite Difference Method (FDM) and many more. The Functional Approximation method consists of Weighted Residual methods and Variational methods. To obtain approximate solutions to problems, weighted residual methods are used for which characteristics are expressed in terms of differential equations [10]. The essential simultaneous equations to find the solution can be achieved from the governing differential equation. The weighted residual methods are a) Point collocation method, b) Sub domain method, c) Least square method and 4) Galerkin's method. We are more focussed on Galerkin's method as it has been implemented in section 1 of this paper.

In Galerkin's method, a trial function which is an approximate function is considered. This function is then substituted in the differential equation to find the Residual 'R' [12]. The boundary conditions must be satisfied by this trial function. The residual is then multiplied by a weighing function and the domain of integral of the product is equated to zero. In Galerkin's method, the trial function only is taken as weighing function.

Apart from Galerkin's method, the other method implemented in this paper is the Finite Element Method (FEM) which is one of the most prominent numerical method. In this method, the complex region defining the domain is divided into smaller elements called finite elements [2]. The physical properties like shape, dimensions and other boundary conditions are imposed on the elements. Then these elements are assembled together and the solution for the entire system can be obtained. FEM requires trial functions and there are infinite number of trial functions that can be used. Actually, these trial functions are grouped into a) Lagrange or C^0 trial function and b) Hermite or C^1 trial functions [2].

2. SCOPE AND OBJECT

A. Application of Galerkin weighted residual method

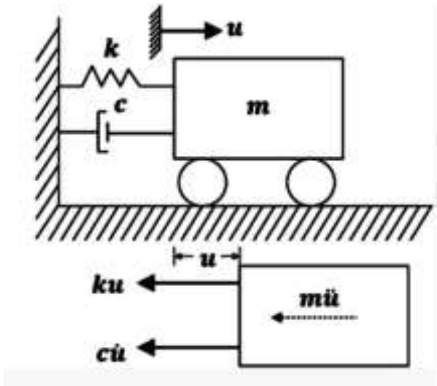
The Galerkin weighted residual method is used for solving the SDOF system equation of motion under the Free and Forced Vibrations. A time discretized solution of this system is required. As this problem requires continuity across displacement and velocity as well as acceleration, higher order trial functions are used to obtain a smooth result. The same problem is solved by analytical method and the results can be plotted together and compared.

B. FEM Formulation with Galerkin Method

"The finite element method (FEM) is a numerical method for resolving problems which are represented by partial differential equations or can be formulated as functional minimization" [10]. A cluster of finite elements is used to represent domain of interest. There are mainly three approaches of Finite Element Formulation i.e. a) Direct approach b) Variational Formulation and c) Galerkin Method. Galerkin method being the most popular approach and appropriate for our problem, is taken up. The problem, after being solved by analytical and Galerkin's Weighted Residual Method is formulated and solved using Galerkin's FEM Method. The results of the solution by this method are plotted and compared with solutions of other approaches.

3. SDOF PROBLEM DEFINITION

The structural system has been modelled as a basic oscillator with viscous damping as can be seen in the following figure. In this figure *m* represents the mass while *k* indicates the spring constant of the oscillator. The viscous damping coefficient is given by *c*.



In the problem taken up by us, a system consisting of a mass (*m*) of = 1 kg and a spring constant (*k*) = 0.025 is viscously damped such that (*C*) = 0.016. When considering the forced vibration case, the harmonic loading *F(t)* is taken as $F_0 sin(0.04\pi t)$.

3.1. ANALYTICAL SOLUTION OF THE PROBLEM

3.1.1 Solution for viscously damped SDOF system under Free Vibration:

Equation of motion: $m\ddot{u} + c\dot{u} + ku = 0$

$\ddot{u} + 0.016\,\dot{u} + 0.025u = 0$

Boundary conditions: $u_0 = 5$, $\dot{u}_0 = 0$

Solution: $u(t)=e^{-(c/2m)t}(A\cos\omega_D t + B\sin\omega_D t)$

where *A* and *B* are integration constants which are later calculated by substituting the initial boundary conditions and ω_D, the damping frequency of the system is,

$$\omega_D = \omega\sqrt{1-\xi^2} \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

$\omega = 0.158$ rad/s

$\xi = C / 2m\omega = 0.0506$

$\omega_D = 0.1578$

Finally, when the initial conditions of displacement and velocity, u_0 and v_0, are introduced, the constants A and B can be calculated and substituted:

$$u(t) = e^{-\xi\omega t}\left(u_0\cos\omega_D t + \frac{v_0 + u_0\xi\omega}{\omega_D}\sin\omega_D t\right)$$

This can also be written as,

$$u(t) = Ce^{-\xi\omega t}\cos(\omega_D t - \alpha)$$

where

$$C=\sqrt{u_0^2+\frac{(v_0+u_0\xi\omega)^2}{\omega_D^2}}$$

and

$$\tan\alpha=\frac{(v_0+u_0\xi\omega)}{\omega_D u_0}$$

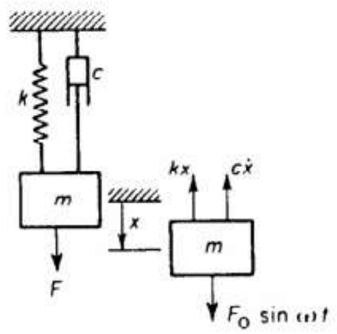
Thus, the solution for Damped Free Vibration of the system is

x[t] = Exp[-0.00800*t] * (5.00*Cos [0.158*t] + (0.253) * Sin [0.158*t]) cm

3.1.2 Solution for viscously damped Forced Vibration with constant harmonic excitation:

$\ddot{v} + 0.016\,\dot{v} + 0.025u = 2\sin(0.04\pi t)$.

Boundary conditions: $u_0 = 0$, $\dot{v}_0=0$



As seen from the figure, the damping force as well as the spring force act in the opposite direction of the motion of the mass. The excitation force acts on the mass with a frequency ω.

$$m\ddot{x}+c\dot{x}+kx=F_o Sin\omega\, t$$

It is a linear non homogeneous second order differential equation. The complete solution will comprise of particular integral as well as complimentary function. The complimentary function of any equation is obtained by equating the equation of complementary function to zero.

The complete solution is:

$$x = x_c + x_p$$

The complementary function solution is given by the following equation

$$x_c = A_2 e^{-\zeta\omega_n t} Sin\left|\sqrt{1-\xi^2}\,\omega_n t+\varphi_2\right|$$

This equation has two constants, which can be found out by applying the boundary conditions. But the boundary conditions cannot be applied to partial solution i.e. only the complimentary function. It is necessary to know the complete response of the system before applying boundary conditions [15]. Thus, particular integral needs to be calculated first.

The particular solution X_p is

$$x_p = XSin(\omega t-\varphi)$$

where

$$X=\frac{X_{st}}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\xi(\omega/\omega_n)]^2}}$$

$$\varphi=\tan^{-1}\left[\frac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right]---9$$

and

$$X_{st}=F_o/k$$

We already have, from the solution of free vibration response, the constants

ω_n = natural frequency = 0.158 rad/s

$\xi = C / 2m\omega = 0.0506$

ω = forcing frequency = 0.04 π rad/s

F_0= 2 N, k= 0.025 N/m

Substituting these values, we get,

X = 212.21

ϕ = 0.215 rad

X_p= 212.21 sin (0.04 π t – 0.215)

$X = X_c + X_p$

$$x=A_2e^{-\zeta\omega_n t}Sin\left|\sqrt{1-\xi^2}\,\omega_n t+\varphi_2\right|+\frac{X_{st}Sin(\omega t-\varphi)}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\xi(\omega/\omega_n)]^2}}$$

Thus, the solution for Forced vibration with constant harmonic excitation is:

$X(t) = A_2e^{-\xi\omega t}\sin(\omega_d t + \phi)+ 212.21 \sin(0.04\pi t - 0.215)$

The two constants A2 and φ2 have to be determined from the boundary conditions. After applying the boundary conditions the values of constants were found to be

A_2 =174.928 and φ_2 = 0.2618

Now this same problem will be solved by Galerkin's weighted Residual method for 0 to 40 seconds.

3.2. GALERKIN SOLUTION OF THE PROBLEM

Weighted Residual Methods begin with an approximation of the solution and require that its weighted average error is narrowed down further or made to vanish [10]. In this method, we work on differential equation of problem

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irrespective of the variational principle [12]. This method comprises of two key steps. Initially an approximate solution is assumed. The selection of approximate solution is done in such a way that it fulfils boundary conditions for φ. The solution that was assumed is then replaced at the differential equation. Since it is only an approximation, the differential equation does not get completely satisfied resulting in an error which is known as a *Residual*. The residual is then equated to zero while integrating the entire domain [12]. This procedure creates a system of algebraic equations. The next step is to solve the system of equations to give approximate solution.

The Galerkin FEM for the solution of a differential equation consists of the following steps [12]:

(1) multiply the differential equation by a weight function ω(x) and form the integral over the whole domain

(2) if required, integrate by parts to reduce the order of the highest order term

(3) select the order of interpolation

(4) evaluate all integrals over each element to set up a system of equations

(5) solve the system of equations

Let's solve the problem by Galerkin approach

3.2.1 Solution for viscously damped SDOF system under Free Vibration

$\ddot{u} + 0.016\,\dot{u} + 0.025u = 0$

Boundary conditions: $u_0 = 5$, $\dot{u}_0=0$

Trial function: A higher order polynomial trial function is taken for better accuracy. After trials it was seen that 4th order polynomial gave a smooth solution.

Trial function: $x\sim(t) = x1+x2t+x3t^2+x4t^3+x5t^4$

@t=0, x~= 5 ➔ x1 = 5

@t=0, dx~/dt = 0 ➔ x2 = 0

Weight function $w(t) = 5+w3t^2+w4t^3+wt^4$

Now in the Galerkin FEM, one lets the weight functions simply be equal to the shape functions

Residue = R = $2x3+6x4+12x5t^2 + 0.016\,(2x3t+3x4t^2+4x5t^3) + 0.025\,(5+x3t^2+x4t^3+x5t^4)$

A system of linear equations is obtained by integrating the product of Weight function and Residue (R) over 0 to 5 and equating it to zero.

This is the p method of getting accurate solution by integrating by parts.

Thus,

Integration (R)*t^2 over 0 to 5 = 0--- (i)

Integration (R)*t^3 over 0 to 5 = 0---(ii)

Integration (R)*t^4 over 0 to 5 = 0--- (iii)

This system of linear equations obtained for this problem over 0 to 5 is

89.89x3 + 974.01x4 +7704.57x5 = -5.2
339.x3 +3902.9x4 + 32087.4x5= -19.53
1359x3+15982x4 + 137595x5 = -78.125

Solving these equations give

X3= -0.0538
X4= -0.00142
X5= 0.000132

Thus, providing us with the solution over 0 to 5

$$\boxed{x\sim(t) = 5 - 0.0538t^2 - 0.00142t^3 + 0.000132t^4}$$

In the same way further approximations are carried out in the of 5 to 10, 10 to 15….(n-5) to n

The solution thus obtained is plotted as follows

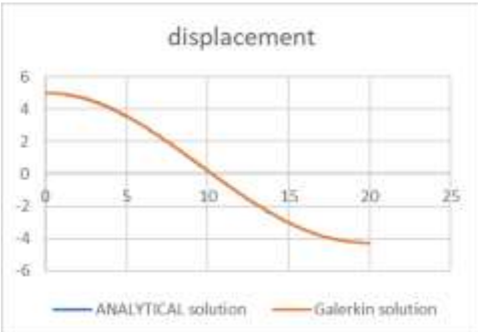

Fig. a) Plot of displacement response by analytical and Galerkin method

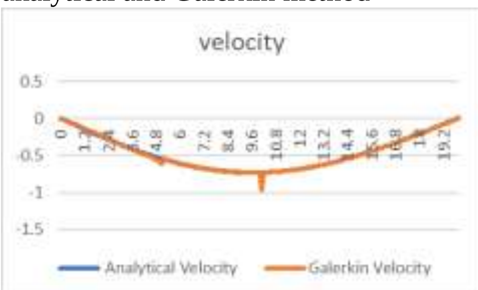

Fig. b) Plot of velocity response by analytical and Galerkin method

Fig. c) Plot of velocity response by analytical and Galerkin method

3.2.2 Solution for viscously damped Forced Vibration with constant harmonic excitation:

$\ddot{v} + 0.016\,\dot{v} + 0.025u = 2\sin(0.04\pi t)$.

Boundary conditions: $u_0 = 0$, $\dot{v}_0=0$

This dynamic forced vibration problem is solved by Galerkin up to 40 seconds and the solution is plotted and compared with analytical solution.

Trial function: A higher order polynomial trial function is taken for better accuracy. After trials it was seen that 4^{th} order polynomial gave a smooth solution.

Trial function: $x\sim(t) = x1+x2t+x3t^2+x4t^3+x5t^4$

@t=0, x~= 0 ➔ x1 = 0

@t=0, dx~/dt = 0 ➔ x2 = 0

Weight function $w(t) = w3t^2+w4t^3+wt^4$

Now in the Galerkin FEM, one lets the weight functions simply be equal to the shape functions

Residue = R = $2x3+6x4+12x5t^2 + 0.016\,(2x3t+3x4t^2+4x5t^3) + 0.025\,(5+x3t^2+x4t^3+x5t^4) - 2\sin(0.04\pi t)$

A system of linear equations is obtained by integrating the product of Weight function and Residue (R) over 0 to 20 and equating it to zero.

Thus,

Integration $(R)*t^2$ over 0 to 20 = 0--- (i)

Integration $(R)*t^3$ over 0 to 20 = 0---(ii)

Integration $(R)*t^4$ over 0 to 20 = 0-- (iii)

This system of linear equations obtained for this problem over 0 to 20 is

22613.3x3+537387x4+712934100x5 = 4481.4

367147x3+8923430x4+219703000x5 = 66209.46

6192760x3+152777000x4+3821310000x5 = 1036707

Solving these equations give

X3= 0.333

X4= 0.02082

X5= -0.0011

Thus, providing us with the solution over 0 to 20

$$\boxed{x\sim(t) = 0.333t^2 + 0.002082t^3 - 0.0011t^4}$$

Similarly, further approximation of 20 to 40 seconds is carried out and the solution is thus plotted for 0 to 40 seconds.

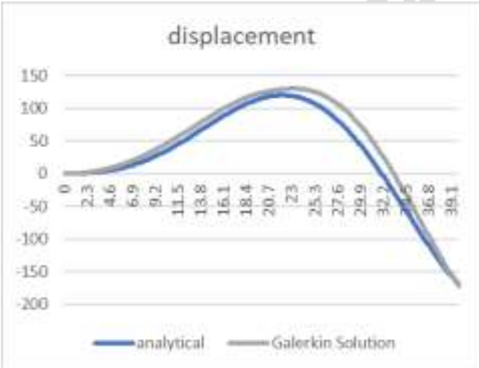

Fig. d) Plot of displacement response by analytical and Galerkin method

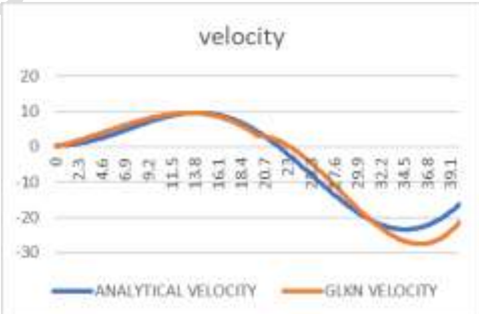

Fig. e) Plot of velocity response by analytical and Galerkin method

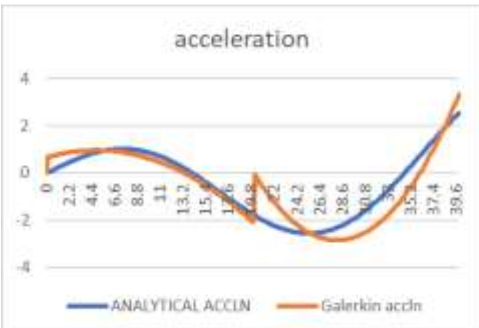

Fig. f) Plot of velocity response by analytical and Galerkin method

It can be seen from the plot of solution by Galerkin that there is an intermittent discontinuity in acceleration and a kink in

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velocity over the intervals. Consequently, to take care of these two anomalies we propose Galerkin FEM Formulation. It is presented in next section of the paper.

4. GALERKIN'S FINITE ELEMENT FORMULATION

The key problem of the weighted residual method is that it's tough to seek out trial functions as a reason of absence previous information of the type of solution [12]. Polynomials that are typically chosen as trial function in such cases may not do a good job of interpolation [12].

FEM is a numerical method for finding solution to the problems which can be represented by partial differential equations. A domain of interest is made up or constitutes of finite elements. "A continuous domain is reworked into a discretized finite number of elements with unknown nodal values" [10]. A system of linear equations ought to be resolved and the nodal values can then be used to determine values of finite elements.

Following are the steps in FEM formulation [12]

a) Discretize the time.

b) Choose interpolation functions

c) Determine the properties of the element

d) Assemble the equations of the element

e) Solve the system of global equations

The foremost necessary steps in finite element method is the choice of actual kind of finite elements. Then comes the characterization of applicable approximating function among the element. This approximating function is called as interpolation polynomial [10].

Several methods are possible to change the problem into discrete F.E. system from its real form. Galerkin method is the most prominent method of formulation when the physical property can be represented as in the form of differential equation. We have used Galerkin method for solving the problem in this paper.

The number of shape functions of element is based upon the count of nodes inside the element. Every shape function is connected to a singular node. Whereas it is zero in rest of the nodes. The nodal worth of each shape function is unity in its own node. Inside the element, shape function's sum is unity. The qualities of shape functions like degree etc is based upon the interpolation polynomial. These properties are valid for all sorts of elements.

Let us take an example of 1d element that is quadratic.

The interpolation polynomial for an element is given as

φ(x) =a1+a2x+a3x2

Where a2, a1 and a3 are constants like co-efficients. Their value is expressed in terms of nodal unknowns. The element should have three nodes because there are three constants in the interpolation polynomial. I, j and k designate the element's nodes. The equivalent nodal values of the unknowns are symbolized by φi, φj, and φk. The nodal coordinates are xp, xq, and xr along with the length of the element Li=xk−xi.

Following expression gives the matrix form of interpolation polynomial

φ(x) =[s]{a}

[s] is given as [s] = [1 x x2]

and the column matrix [a] is

$$[a] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_k \end{bmatrix} = \begin{bmatrix} 1 & x_i & x_i^2 \\ 1 & x_j & x_j^2 \\ 1 & x_k & x_k^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The unknown parameters {a} can be found out by pre multiplying the above equation with [G]−1

$$\begin{aligned} \varphi(x) &= \frac{2}{L_i^2}(x-x_j)(x-x_k)\varphi_i + -\frac{4}{L_i^2}(x-x_i)(x-x_k)\varphi_j + \frac{2}{L_i^2}(x-x_i)(x-x_j)\varphi_k \\ &= N_i\varphi_i + N_j\varphi_j + N_k\varphi_k \\ &= [\mathrm{N}][\varphi] \end{aligned}$$

The shape functions are then given by

$$N_i = \frac{2}{L_i^2}(x-x_j)(x-x_k)$$

$$N_j = -\frac{4}{L_i^2}(x-x_i)(x-x_k)$$

$$N_k = \frac{2}{L_i^2}(x-x_i)(x-x_j)$$

In our problem we have used Quintic Hermite Trial Functions

The purpose of introducing the Quintic Hermite element is to make sure that the second derivatives of the trial function are continuous at the nodes [1]. To this end, consider six unknowns, the two values of p at the end nodes, two values of p at the end nodes and the two values of p'' at end nodes.

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With six unknowns, one can use the Quintic trial function

p(t) = x1+x2t+x3t2+x4t3+x5t4+x 6t5

Rewriting the trial functions concerning the unknown nodal values and expressing shape functions concerning local coordinates, one reaches at [1]

$$
\begin{aligned}
H_1 &= (1 - 10\xi^3 + 15\xi^4 - 6\xi^5) \\
H_2 &= h(\xi - 6\xi^3 + 8\xi^4 - 3\xi^5) \\
H_3 &= h^2(0.5\xi^2 - 1.5\xi^3 + 1.5\xi^4 - 0.5\xi^5) \\
H_4 &= h^2(0.5\xi^3 - \xi^4 + 0.5\xi^5) \\
H_5 &= (10\xi^3 - 15\xi^4 + 6\xi^5) \\
H_6 &= h(-4\xi^3 + 7\xi^4 - 3\xi^5)
\end{aligned}
$$
$$\text{with } H_1(\xi) = H_5(1-\xi),\ H_2(\xi) = -H_6(1-\xi),\ H_3(\xi) = H_4(1-\xi)$$

Substitute these shape functions in finite element equations.

The second derivatives of the shape functions are non-zero for the Quintic Hermite trial function, and so the solution is available using the weighted residual method with no integration by parts. The coefficient matrix will not be symmetric. In cases like this, when shifting to local coordinates, one must use [2]

$$\frac{dN_i}{dx} = \frac{dN_i}{d\xi}\frac{d\xi}{dx}$$
$$\frac{d}{dx}\left(\frac{dN_i}{dx}\right) = \frac{d}{dx}\left(\frac{dN_i}{d\xi}\frac{d\xi}{dx}\right)$$
$$= \frac{d}{dx}\left(\frac{dN_i}{d\xi}\right)\frac{d\xi}{dx} + \frac{dN_i}{d\xi}\frac{d^2\xi}{dx^2}$$
$$= \frac{d^2N_i}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 + \frac{dN_i}{d\xi}\frac{d^2\xi}{dx^2}$$

They need to be connected together or grouped to portray the integrated behavior of the system after the individual element equations are derived. This procedure is governed by the idea of continuity. To find the unknown values or derivatives, the solutions individually obtained in elements at continuous nodes are matched at their common nodes which makes the total solution continuous. Following equation gives the form of the global finite element equation

[K][φ] = [f]

where [K] is the global stiffness matrix and [f] is the global force vector.

Following are the graphs plotted for the solution by GFEM and analytical method.

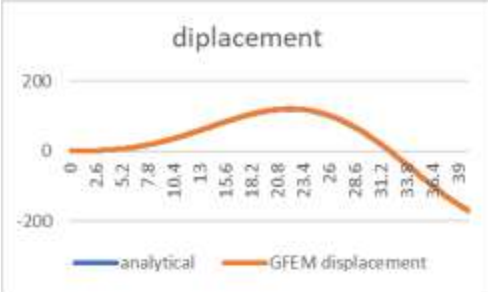

Fig. g) Plot of displacement response by analytical and Galerkin FEM method

Fig. h) Plot of velocity response by analytical and Galerkin FEM method

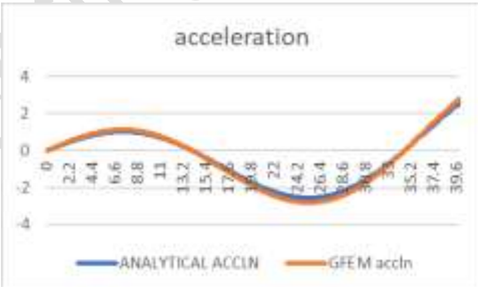

Fig. i) Plot of acceleration response by analytical and Galerkin FEM method

5. RESULTS AND CONCLUSION

A problem of SDOF system with viscous damping was taken up and solved for free vibration as well as forced vibration with constant harmonic excitation. The problem was first completely solved by analytical method to have a benchmark for comparing solutions obtained through other approaches. Analytical solution is based on theoretical mathematical expressions and give exact solution. This problem was then solved by using Galerkin's weighted residual method. The system of equation taken up for problem is the equation of motion and thus demands continuity across displacement, velocity as well as acceleration. The Galerkin method was thus formulated keeping in mind the requirement of 3 DOFs at each end. 4th order polynomial was thus taken up as trial function

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and the solution was obtained. After observing the graphs of displacement, velocity and acceleration by Galerkin method for free and forced vibrations, it was seen that there was an intermittent discontinuity in the acceleration and a kink in velocity. This solution thus needed to be discarded. Consequently, Galerkin's FEM formulation was proposed to solve the problem. To fulfill the demand of continuity across second derivative (C2 continuity) and meet the requirement of 3 DOFs at each end, Quintic Shape functions were introduced. As seen from the graphs of GFEM displacement, velocity and acceleration, this gave a more accurate and acceptable solution

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